Generalized Integrated Square Error Criterion for Optimal and Suboptimal Control Design

Tuncer Miski*
DFVLR Forschungszentrum, Braunschweig, F.R. Germany

By showing the equivalence of quadratic cost functionals to the classical generalized integrated square error (GISE) performance index a simple optimal control design procedure for linear time-invariant single input systems is presented. The eigenvalues of the optimal system can be directly determined from the weighting factors of the GISE. The method closely relates the inverse problem of optimal control theory and gives a new, straightforward solution. It is also shown that the GISE represents a criterion of least-square approximation which can be used to solve the model following problem and to establish a suboptimal control as well. The parameters of a suboptimal control law can be determined by minimizing the GISE. For this purpose, a new simple method of evaluating the GISE is given. The technique is illustrated by designing an aircraft altitude control system.

Introduction

IN the last ten years several studies have been devoted to the inverse regulator problem, as stated by Kalman,[1] "given a control law, find all performance indices for which this law is optimal." The need for a solution was soon recognized since it was laborious to find a proper performance index that led to a satisfying optimal control law. That is, the time responses of the optimal system responses are acceptable and, from the implementation point of view, the control law does not require great actuator deflections.

The problem is now solved for the linear time-invariant systems with an infinite control interval. Chang[2] showed by means of his "root square locus" method that the eigenvalues of a single input system which is optimal with respect to a quadratic cost functional can be determined directly from the terms of this cost functional. Rynaski[3] extended this method to multiple-input systems. The same results were obtained by Kalman[1] for single input and by Anderson[4,5] for multiple-input systems in a different manner. They also pointed out some frequency-domain properties of optimal systems. A very important conclusion to their investigations is the fact that optimality implies sensitivity reduction. All these methods were derived on the basis of the calculus of variations. In contrast to these contributions, Rediess[6] presented a geometrical concept for specifying a "Model Performance Index" (MPI) in terms of a given model.

In earlier publications,[7,8] Krassowski showed how to determine the weighting factors of a generalized integral error (GISE) for matching the system output with a desired transient response. This approach, however, supposes special initial conditions; furthermore, the GISE does not explicitly include the control law. The purpose of this paper is to show that these limitations are of no importance to the optimal single input control problem with infinite control interval. Thus, by modifying Krassowski's method a simple solution of the inverse problem can be given without applying the calculus of variations.

It will be shown that: 1) the GISE is equivalent to any admissible quadratic cost functional; 2) the GISE is equivalent to the MPI which follows from 1; and 3) the weighting factors of the GISE represent the coefficients of a nonnegative polynomial which is identical to the polynomial used in the root squared locus method. The spectral factorization of this polynomial yields the polynomials of the optimal system and its dual.

Postulates 1 and 3 provide a new, simple method of finding the optimal control law and the "inverse" solution of the inverse problem as well. From 3 it follows that the GISE represents a least square approximation criterion which will be proved different in the frequency-domain from Rediess' method. This property leads to a very important application of the GISE. If some state variables are inaccessible a suboptimal control can be established by minimizing the GISE or the MPI, respectively, as they are equivalent to the associated quadratic cost functional.

Both performance indices are evaluated very easily in the frequency-domain by means of the residue theorem; only a set of n linear equations needs to be solved. This algorithm is more efficient than those given in previous literature. Thus, the GISE becomes attractive for parameter optimization of control systems, e.g., sub-optimal design procedure as suggested here.

Analytic Development

Consider the linear time-invariant system in phase-variable canonical form

$$\dot{x}(t) = Ax(t) + bu(t) \tag{1}$$

where $\mathbf{x}$ is the n-dimensional state vector; the initial conditions (IC's) are given with $\mathbf{x}(0)$; $u(t)$ is the scalar input; and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & & \mathbf{I}_{n-1} \\ -a_1 & -a_2 & \cdots & -a_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The a_i are coefficients of the characteristic polynomial of the open-loop system. $\mathbf{I}_k$ denotes a unit matrix of kth order.

The basic optimal problem is to find a control law u^+ which minimizes the quadratic cost functional

$$V(u,x(0)) = \int_0^\infty (\|x\|_Q^2 + ru^2)dt \tag{2}$$

To ensure the stability of the closed-loop system it is assumed that a) the pair $(\mathbf{A},\mathbf{b})$ is completely controllable, b) the pair $(\mathbf{A},\mathbf{Q}^{1/2})$ is completely observable.

Received February 9, 1973; revision received November 19, 1973.
Index category: Aircraft Handling, Stability, and Control.

*Research Engineer, Institut für Flugführung, Flight Control Branch. Presently Instructor, Dept. of Mechanical Engineering, Middle East Technical University, Gaziantep, Turkey.

The Equivalence

The well-known version of the generalized integral criterion suggested by Feldbaum[9] is the

$$\text{GISE} = \int_0^\infty \|\tilde{x}\|_C^2 dt \tag{3}$$

with the diagonal weighting matrix

$$C = \begin{bmatrix} c_1^2 & & & 0 \\ & c_2^2 & & \\ & & \ddots & \\ 0 & & & c_{n+1}^2 \end{bmatrix}$$

To show the equivalence of the GISE to the cost functional Eq. (2) the control u will be eliminated by

$$u(t) = \tilde{a}'\tilde{x}(t) \tag{4}$$

where $\tilde{x}$ is defined as the extended state vector

$$\tilde{x}' = [x' \vdots x_{n+1}] \tag{5}$$

and the new expressions

$$x_{n+1} = \dot{x}_n$$

$$\tilde{a}' = [a \vdots 1], \text{ with } a = [a_1, a_2, \cdots, a_n]$$

are introduced. Substituting Eq. (4) into Eq. (2) yields

$$V(u,x(0)) = V(\tilde{x}) = \int_0^\infty \|\tilde{x}\|^2_{\tilde{Q}} dt \tag{6}$$

where

$$\tilde{Q} = \begin{bmatrix} Q + raa' & ra \\ ra' & r \end{bmatrix} \tag{6a}$$

By eliminating u, the cost functional Eq. (2) is carried over to Eq. (6), which depends on the $\tilde{x}$ in the same way as the GISE in Eq. (3). The only difference is that $\tilde{Q}$ is a nondiagonal matrix. In order to derive a connection between $\tilde{Q}$ and $\tilde{C}$ the integrand of Eq. (6) can be written as

$$\|\tilde{x}\|^2_{\tilde{Q}} = \|\tilde{x}\|^2_{\text{diag } \tilde{Q}+K} + \frac{d}{dt} \|x\|^2_L \tag{7}$$

where the diagonal $(n+1) \times (n+1)$ matrix K and $n \times n$ matrix L can be obtained, as follows:

The last term of Eq. (7) is given by

$$\frac{d}{dt} \|x\|^2_L = \dot{x}'Lx + x'L\dot{x} \tag{8}$$

Recalling Eq. (1) and Eq. (5), and introducing T_1 and T_2 by

$$\dot{x} = [0 \vdots I_n]\tilde{x} = T_1\tilde{x}$$

$$x = [I_n \vdots 0]\tilde{x} = T_2\tilde{x} \tag{9}$$

Eq. (8) can be rewritten

$$\frac{d}{dt} \|x\|^2_L = \|\tilde{x}\|^2_{T_1'LT_2 + T_2'LT_1} \tag{10}$$

Finally, applying this to Eq. (7) results in

$$\tilde{Q} = \text{diag } \tilde{Q} + K + T_1'LT_2 + T_2'LT_1 \tag{11}$$

from which K and L can easily be determined. Equation (7) will be integrated to obtain

$$V(\tilde{x}) = \int_0^\infty \|\tilde{x}\|^2_{\text{diag } \tilde{Q}+K} dt + \|x\|_L^2 /_0^\infty \tag{12}$$

Recalling the assumption of the stability

$$\lim_{t\to\infty} \|x\|^2_L = 0$$

and letting

$$C = \text{diag } \tilde{Q} + K \tag{13}$$

the cost functional Eq. (2) becomes

$$V(u,x(0)) = \int_0^\infty \|\tilde{x}\|_C dt - \|x(0)\|_L^2 \tag{14}$$

Since the extremum of V does not depend on the quadratic form of the IC's the equivalence is proved, i.e., Eqs. (2) and (3) lead to the same control law. To construct the GISE only the matrices $\tilde{Q}$ and $\tilde{K}$ are needed. $\tilde{Q}$ is given by Eq. (6a), and K can be determined from

$$k_{11} = k_{n+1,n+1} = 0$$

$$k_{ii} = \sum_{j=1}^{\min(n+1-i,i-1)} (-1)^j \tilde{q}_{i+j}\tilde{q}_{i-j} \tag{15}$$

which is derived from Eq. (11).

The Relation between Weighting Factors and the Eigenvalues

A similar proof of the equivalence of the GISE to the Integral

$$J(\tilde{x}) = \int_0^\infty \|\tilde{x}\|^2_{dd'} dt \tag{16}$$

with

$$d' = [d_1, d_2, \ldots, d_{n+1}]$$

can be given since the dyad dd' is a special form of $\tilde{Q}$. The procedure of the previous Section leads then to

$$\text{GISE} = J(\tilde{x}) + \|x(0)\|^2_H \tag{17}$$

with

$$c_i^2 = \sum_{j=1}^{2i} (-1)^{i+j} d_j d_{2i-j} \tag{18}$$

for $i = 1,2,\ldots n+1$, and $d_j = 0$ for $j > n$. The integrand of I

$$\|\tilde{x}\|^2_{dd'} = (d'\tilde{x})^2$$

is non-negative definite. Then the minimum of the GISE is given by

$$\text{GISE} = \|x(0)\|^2_H$$

for

$$d'\tilde{x}(t) = 0 \tag{19}$$

which represents the homogenous differential equations of the extremal.

Considering high-order systems, there is no explicit solution to Eq. (18) for d_i. But the determination of d_i can be achieved by utilizing the fact that the polynomial

$$C(s^2) = c_1^2 - c_2^2 s^2 \pm \ldots (-1)^{n+1} c_{n+1}^2 s^{2n}$$

is a product of the polynomials $D(s)$ and $D(-s)$.

$$C(s^2) = D(s)D(-s) \tag{20}$$

where $D(s)$ denotes the characteristic equation of the closed-loop system with

$$D(s) = d_1 + d_2 s + \ldots + d_{n+1} s^2 \tag{20a}$$

To determine d_i the polynomial $D(s)$ can now be constructed from n negative real-part (stable) zeros s_i of $C(s^2)$

$$D(s) = c_{n+1} \prod_{i=1}^{n} (s - s_i)$$

Hence, the characteristic equation of the closed-loop system is directly derived from the weighting factors c_i^2 of the GISE. It is worthwhile to note that the polynomial $C(s^2)$ is identical to the determinant of the canonical equations of the optimal system as used by Rynaski.[3]

To solve the actual inverse problem the procedure can now be applied in reverse order. The weighting factors can be calculated from the desired d_i by Eq. (18); and by eliminating $x_{n+1}(t)$ with respect to the relationships in Eqs. (4) and (5). The GISE can be reduced to an ordinary quadratic cost functional.

Note that a direct transformation of the cost functional into the integral $J(\tilde{x})$ is possible. But for this a complicated system of quadratic equations is to be solved. The derivation of the weighting factors of the GISE from Eq. (15) is considerably easier.

Properties of the GISE

As already shown in Eq. (16), the GISE is equivalent to the integral $J(\tilde{x})$ which is proportional to the model performance index

$$\text{MPI} = \frac{1}{\|\alpha\|^2} \int_0^\infty \|\tilde{x}\|^2_{\alpha\alpha'} dt \tag{21}$$

Rediess[5] shows that the MPI is a generalized measure of the distance between the system's time response trajectory and the characteristic plane of a model $\boldsymbol{x}_M$ given by

$$\alpha' \tilde{x}(t) = 0 \tag{22}$$

where the elements of α are the coefficients of the model's characteristic equations [note the similarity between Eqs. (19) and (22)]. Therefore, a system which minimizes the MPI or the equivalent GISE, respectively, approximates the model. This can also be shown in the frequency-domain which brings new insight into this concept.

By applying Parseval's theorem to Eqs. (16)

$$J(\tilde{x}) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \langle d'\tilde{X}(s) \rangle^2 ds,$$

$$\text{where} \langle d'\tilde{X}(s) \rangle^2 = \tilde{X}'(-s) dd' \tilde{X}(s)$$

a sufficient and necessary condition for the lower value of J may be established in the frequency-domain

$$d'\tilde{X}(s) = 0 \tag{23}$$

This corresponds to the condition in Eq. (19). Assuming that the model in Eq. (22) satisfied Eq. (23), i.e., $\alpha = \boldsymbol{d}$, the first model state variable can be obtained

$$X_{M_1} = \frac{Z(s, \boldsymbol{x}(0))}{D(s)} \tag{24}$$

where

$$\begin{aligned} Z_1(s, \boldsymbol{x}(0)) &= d_1 x_1(0) + d_2 x_2(0) + \ldots + d_n x_n(0) \\ &+ (d_2 x_2(0) + \ldots + d_n x_{n-1}(0)) s \\ &+ \ldots + d_n x_1(0) s^{n-1} \end{aligned} \tag{25}$$

Taking the Laplace transform of $\boldsymbol{d}'\tilde{\boldsymbol{x}}(t)$

$$\boldsymbol{d}'\tilde{X}(s) = D(s) X_1(s) - Z(s, \boldsymbol{x}(0)) \tag{26}$$

and substituting Eq. (24) into Eq. (26) yields

$$\boldsymbol{d}'\tilde{X}(s) = D(s)(X_1(s) - X_{M_1}(s)) \tag{27}$$

Thus, the integral

$$J(\tilde{x}) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \langle D(s)(\tilde{X}_1(s) - X_{M_1}(s)) \rangle^2 ds \tag{28}$$

and the equivalent GISE are identical with the criterion of a least-square approximation in the frequency-domain, where the error is weighted by the characteristic equation of the model. This is also valid for models of lower order.

Suboptimal Control Design

It is well known that the optimal control law depends on all of the state variables, and it can not be reached, if some of the state variables are inaccessible. But, as shown in the previous section a GISE which is equivalent to V can be though of a least-square approximation criterion. Thus, a suboptimal control can be found, if the GISE is minimized subject to the corresponding gains of the available state variables. The design procedure is then: 1) transform the system into the phase-variable canonical form in Eq. (1) if the system is not already given in this form; 2) determine $\tilde{Q}$ from Eq. (6), and $\tilde{C}$ from Eqs. (13) and (15), respectively; 3) find the gains of the available state variables which minimize the GISE with the weighting matrix, $\boldsymbol{C}$ through a parameter optimization.

Although the procedure is very simple, there is one drawback. The suboptimal control depends on the IC's since the integrand in Eq. (28) generally does not vanish for the suboptimal control. However, it is possible to define an average of performance for a set of random IC's which can be treated then also as an ordinary quadratic cost functional.

A very important problem of the numerical optimization is the evaluation of the function which will be minimized. The frequency-domain representation leads to a more efficient method of evaluating the GISE than those given in previous literature.

Taking the Laplace transform of each member of the GISE in Eq. (3) it can be rewritten

$$\text{GISE} = \sum_{i=1}^{n+1} C_i^2 J^i \tag{29}$$

where

$$J^i(\boldsymbol{x}(0)) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} X_i(-s) X_i(s) ds$$

by assuming

$$X_i(s) = F^i(s)/E(s) \tag{30}$$

where $E(s)$ is the characteristic equation of the closed-loop systems; and

$$F^{I}(s)=sF^{i-1}-x^{i-1}(0)E \quad (31)$$

the J^i can be determined (Ref. 10)

$$J^i=(-1)^{n-1}c_i^2\frac{k_n^i}{2e_{n+1}} \text{ for } i=1,2,...,n+1$$

Here k_n^i denotes the nth element of the vector $\boldsymbol{k}^i$ which is given by

$$\boldsymbol{H}_n\boldsymbol{k}^i=\boldsymbol{l}^i \quad (32)$$

where $\boldsymbol{H}_n$ is the Hurwitz determinant of the system; and

$$l_\mu^i=\sum_{\nu=1}^{2\mu}(-1)^{\nu+\mu}f_\nu^i f_{2\mu-\nu}^i, f_\nu=0 \text{ for } \nu>n \quad (32a)$$

Since $\boldsymbol{H}_n$ does not depend on i, Eqs. (29)-(32) give

$$\text{GISE}=\frac{(-1)^{n-1}}{2e_{n+1}}[\boldsymbol{H}_n^{-1}]_n\sum_{i=1}^{n+1}c_i^2\boldsymbol{l}^i \quad (33)$$

where $[\boldsymbol{H}_n^{-1}]_n$ denotes the nth row of $\boldsymbol{H}_n^{-1}$. Here, only a system of n linear equations has to be solved, which requires approximately $n^3/3$ multiplications. Therefore, the GISE can be evaluated at an effort of $7n^3/12$ multiplications, including $n^3/4$ multiplications for the determination of $\boldsymbol{l}^i$ from Eq. (32a). As shown in Table 1, this procedure is superior to the existing methods.

Suboptimal Flight Control System

The linearized longitudinal equations of a HFB-320-type aircraft in a approach flight condition, can approximately be given by

$$\dot{z}(t)=Fz(t)+g(t) \quad (34)$$

where

$$\boldsymbol{F}=\begin{bmatrix}0 & 1 & 0 & 0 & 0\\ -1.3 & -1.028 & 0.026 & 1.303 & 0\\ -0.086 & 0 & -0.06 & -0.079 & 0\\ 0.769 & 0.019 & 0.336 & -0.778 & 0\\ 0 & 0 & 0 & 23.753 & 0\end{bmatrix}, \boldsymbol{g}=\begin{bmatrix}0\\ -2.56\\ 0\\ 0.07\\ 0\end{bmatrix}$$

the state variables are, z_1 = pitch altitude, rad; z_2 = pitch rate, rad; z_3 = incremental air speed (nondimensional), z_4 = incremental angle of flight path, rad, z_5 = altitude (nondimensional), and the control variable $u(t)$ = incremental elevator angle, rad. The actual values of the airspeed v and the altitude h are given by $v=57.6\, z_s$ (m/sec) and $h=4.425\, z_5$ (m), respectively. Choosing $\boldsymbol{Q}=\boldsymbol{I}_5$ and $r=18.000$ the quadratic cost functional is formulated according to Eq. (2).

For finding the control law by means of an equivalent GISE the system of Eq. (34) will be transformed into a phase-variable canonical form

$$\boldsymbol{x}=\boldsymbol{T}\boldsymbol{z} \quad (35)$$

where

$$\boldsymbol{T}=\begin{bmatrix}-0.163 & -0.507 & -159.386 & -18.733 & -1.1\\ -0.011 & -0.011 & 3.244 & 0.389 & 0\\ 0.006 & -0.015 & -0.063 & -0.546 & 0\\ 0.395 & -0.011 & 0.180 & 0.410 & 0\\ 0.317 & -0.399 & 0.149 & -0.291 & 0\end{bmatrix}$$

Thus the quadratic cost functional becomes

$$V(u,x(0))=\int_0^\infty(\|x\|^2_{(T^{-1})'T}+18{,}000u^2)\text{d}t \quad (36)$$

Table 1 Evaluation of the GISE

Ref.	Method	Equations	Multiplications
11	via 2nd method of Lyapunov	$n(n+1)/2$	$(n^2+n)^3/24$
12	Numerical integration of a matrix differential eq.	not analytic	$2n^3$ each integration step
—	via residue theorem	n	$7n^3/12$

Corresponding to Eq. (4), the control law can be expressed now by the extended state vector

$$u=[0,\ 0.08,\ 0.16,\ 2.21,\ 1.87,\ 1.0,\ 1.0]\tilde{x} \quad (37)$$

Referring to Eqs. (6a), and (13) and (15), respectively, Eq. (33) leads to the equivalent GISE with the weighting matrix

$$\boldsymbol{C}=\begin{bmatrix}0.8 & & & & & \\ & 2089.1 & & & & \\ & & -5422.7 & & & \\ & & & 79749.1 & & \\ & & & & -14949.1 & \\ & & & & & 18{,}000\end{bmatrix} \quad (38)$$

Then, the eigenvalues of the optimal system can be determined as the left-hand plane zeros of the polynomial $C(s^2)$ as defined in (20),

$$\begin{aligned}s_1 &= -0.002\\ s_{2,3} &= -0.91\pm 1.12j\\ s_{4,5} &= -0.26\pm 0.31j\end{aligned} \quad (39)$$

Thus the quadratic cost functional Eq. (32), can be interpreted as a least-square approximation criterion of a model given by the poles of Eq. (39).

Let $\boldsymbol{k}$ denote the constant feedback vector:

$$u^+(t)=\boldsymbol{k}'z(t) \quad (40)$$

and recall the equations of the closed-loop system

$$\dot{x}(t)=[A+bk'T^{-1}]x(t) \quad (41)$$

Then $\boldsymbol{k}$ is found by equating the last row elements of $(A+bk'T^{-1})$ with the corresponding coefficients of a characteristic polynominal of the closed-loop system. The last can be determined referring to Eq. (19):

$$D(s)=0.007+0.34\,s+1.43\,s^2+3.22\,s^3+2.35\,s^4+s^5$$

which yields

$$k'=[0.25,\ 0.2,\ 0.33,\ 0.43,\ 0.43,\ 0.008]$$

As shown in Fig. 1, this control law achieves an acceptable response to an initial altitude error and requires quite reasonable magnitudes of elevator deflection (Fig. 2.).

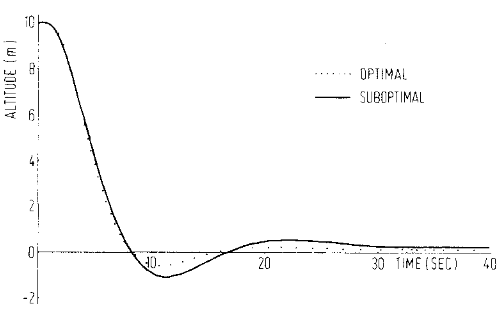

Fig. 1 Aircraft response to a 10m initial altitude deviation.

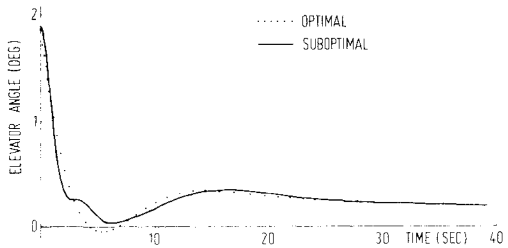

Fig. 2 Control input response to a 10m initial altitude deviation.

Since the angle of flight path, z_4, is not directly measurable an observer must be implemented to realize the optimal control law. This may not be economically feasible. Therefore, a suboptimal control can be established by defining a partial feedback

$$\boldsymbol{k}'=[k_1,k_2,k_3,0,k_5]$$

and by minimizing the GISE, with the weighting matrix $\boldsymbol{C}$ in Eq. (38), with respect to $\boldsymbol{k}$.

To evaluate the GISE from Eq. (33) the closed-loop system must be represented in the frequency domain. By assuming the IC's $\boldsymbol{x}'(0)=[0,\ldots,0,h_0]$, the $X_1(s)$ is given by

$$X_1(s)=h_0\,\frac{e_1+e_2s+\ldots+s^4}{e_1+e_2s+\ldots+s^5}$$

where $e_i=-a_i+\boldsymbol{k}'(\boldsymbol{T}^{-1})_i$, and $(\boldsymbol{T}^{-1})_i$ denotes the ith column of $\boldsymbol{T}^{-1}$. Thus the GISE can be expressed now by the elements of the partial feedback, such that the parameter optimization yields the suboptimal control law

$$\boldsymbol{k}=[0.62,-0.013,\ 0.48,\ 0,\ 0.008]$$

Even though the corresponding response (Fig. 1) indicates a greater overshoot than the optimal response it can still be acceptable regarding the expense for an observer implementation.

Conclusion

The equivalence of the GISE to the quadratic cost functionals is shown. This leads to a simple, straightforward solution of the inverse problem of optimal control theory. By means of polynomial algebra, the eigenvalues of the optimal closed-loop system can be determined directly from the weighting factors of the GISE. Inversely, the weighting factors can be obtained, when the eigenvalues satisfying some engineering specifications are given.

It is shown that GISE represents a criterion of the least-square approximation. Using this property a method of finding the suboptimal control via parameter optimization is presented.

A new method of evaluating the GISE, which is an important factor for a numerical optimization, is proposed. The method is simple and computationally superior to those given in previous literature. The given example shows that a suboptimal control law can still yield acceptable solutions so that a control system can be designed without an observer.

References

[1] Kalman, R. E., "When is a Linear Control System Optimal?" *Transactions of the ASME, Journal of Basic Engineering*, Ser. D, Vol. 86, March 1964, pp. 55-60.

[2] Chang, S. S. L., *Synthesis of Optimum Control Systems*, McGraw-Hill, New York, 1961.

[3] Rynaski, E. G., "The Multivariable Root-Square-Locus," *An Optimal System Design Aid, AIAS/ION Guidance and Control Conference*, Minneapolis, Minn., 1965, p. 124.

[4] Anderson, B. D. O., "The Inverse Problem of Optimal Control," Tec. Rept. 6560-3, SEL-66-038, April 1966,Stanford Electronics Lab., Stanford, Calif.

[5] Anderson, B. D. O. and Moore, J. B., *Linear Optimal Control*, Prentice-Hall, Englewood Cliffs, N.J., 1971.

[6] Rediess, H. A., "A New Model Performance Index for Engineering Design of Flight Control Systems." *Journal of Aircraft*, Vol. 7., Nov.-Dec. 1970, pp. 542-549.

[7] Krassowski, A. A., *Performance Indices for the Control Processes*, Maschgis, Moscow, 1949.

[8] Solodovnikow, V. V., *Foundations of the Servomechanism*, Oldenbourg Verlag, Munich, 1959, Chap. 20 (German edition).

[9] Feldbaum, A. A., "Performance Indices for Control," *Avtomatica i Telemechanica*, Vol. 9, 1948, pp. 3-19.

[10] Drenick, R. F., *Die Optimierung linearer Regelsysteme*, Oldenbourg Verlag, Munich, 1967.

[11] MacFarlane, A. G. J., "The Calculation of Functionals of the Time and Frequency Response of a Linear Constant Coefficient Dynamical System," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 16, Pt. 2, 1963, pp. 260-271.

[12] Rediess, H. A., "A New Model Performance Index for the Engineering Design of Control Systems," Ph. D. Thesis, Dept. of Aeronautics and Astronautics MIT, Cambridge, Mass., Dec. 1968.